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# Journal of Mathematical Analysis and Applications

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## A geometric characterization of some biholomorphic mappings in $\mathbb{C}^n$

Piotr Liczberski

Institute of Mathematics, Technical University of Łódź, Ul. Żwirki 36, 90-924 Łódź, Poland

### ARTICLE INFO

#### Article history:

Received 16 March 2010

Available online 29 September 2010

Submitted by Richard M. Aron

#### Keywords:

Biholomorphic mappings

Locally biholomorphic mappings

Strongly starlike mappings of order alpha

### ABSTRACT

In the paper an internal geometric characterization of strongly starlike mappings of order alpha in  $\mathbb{C}^n$  is given.

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### 1. Introduction

Let  $\mathbb{B}^n$  denote the open unit ball  $\{z \in \mathbb{C}^n: \|z\|^2 = \langle z, z \rangle < 1\}$ , where  $\langle z, w \rangle$  denotes the Euclidean inner product of vectors  $z, w \in \mathbb{C}^n$ .

**Definition.** (See [7], compare also [1,15].) A locally biholomorphic mapping  $f: \mathbb{B}^n \rightarrow \mathbb{C}^n$  such that  $f(0) = 0$ ,  $Df(0) = I$ , is said to be strongly starlike of order  $\alpha \in (0, 1]$ , if

$$|\arg((Df(z))^{-1} f(z), z)| < \alpha \frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}. \quad (1)$$

Since relation (1) implies

$$\operatorname{Re}((Df(z))^{-1} f(z), z) > 0, \quad z \in \mathbb{B}^n \setminus \{0\}, \quad (2)$$

every strongly starlike mapping  $f$  of order alpha is also starlike (see the well-known theorem of Matsuno [12], Suffridge [16] and Kikuchi [6]). Hence every strongly starlike mapping  $f$  of order alpha is biholomorphic and satisfies the condition

$$tf(\mathbb{B}^n) \subset f(\mathbb{B}^n), \quad t \in [0, 1].$$

Strong starlikeness of order alpha can be also characterized in terms of spirallikeness relative to a linear operator. A holomorphic mapping  $f: \mathbb{B}^n \rightarrow \mathbb{C}^n$  such that  $f(0) = 0$ ,  $Df(0) = I$ , is said to be spirallike relative to a linear operator  $A$ , if  $f$  is biholomorphic and fulfils the condition

$$e^{tA} f(\mathbb{B}^n) \subset f(\mathbb{B}^n), \quad t \in (-\infty, 0),$$

where

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

E-mail address: [piliczb@p.lodz.pl](mailto:piliczb@p.lodz.pl).

For spirallikeness in  $\mathbb{C}^n$  and in Banach spaces, see for instance the paper of Elin, Reich and Shoikhet [2], Gurganus [5], and Suffridge [18].

G. Kohr and the author proved in [7] that a mapping  $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$  is strongly starlike of order  $\alpha \in (0, 1]$  iff  $f$  is spirallike relative to operator  $A = e^{i\beta}I$  for every  $\beta \in [(\alpha - 1)\frac{\pi}{2}, (1 - \alpha)\frac{\pi}{2}]$ .

In the present paper we will give another characterization of strong starlikeness of order  $\alpha \in (0, 1)$ .

## 2. Main result

By  $\mathbb{E}_\alpha$  for  $\alpha \in (0, 1)$ , let us denote a planar closed lens bounded by a Jordan curve  $\Gamma_\alpha$  with the following polar coordinates equation:

$$\zeta = \rho_\alpha(\theta)e^{i\theta}, \quad \theta \in \left[(\alpha - 1)\frac{\pi}{2}, (1 - \alpha)\frac{\pi}{2}\right],$$

where

$$\rho_\alpha(\theta) = \begin{cases} (\cos(\alpha\frac{\pi}{2}))^{-1} \cos(\theta - \alpha\frac{\pi}{2}), & \theta \in [(\alpha - 1)\frac{\pi}{2}, 0], \\ (\cos(\alpha\frac{\pi}{2}))^{-1} \cos(\theta + \alpha\frac{\pi}{2}), & \theta \in [0, (1 - \alpha)\frac{\pi}{2}]. \end{cases} \quad (3)$$

Below we give the main theorem which includes the announced characterization of strongly starlike mappings of order  $\alpha \in (0, 1)$ .

**Theorem.** Let  $\alpha \in (0, 1)$  and  $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$  be a locally biholomorphic mapping normalized by the conditions  $f(0) = 0$ ,  $Df(0) = I$ . Then the following statements are equivalent:

- (i)  $f$  is a strongly starlike mapping of order  $\alpha \in (0, 1)$ ;
- (ii)  $f$  is biholomorphic and for every  $w \in f(\mathbb{B}^n)$  there holds

$$w\mathbb{E}_\alpha \subset f(\mathbb{B}^n). \quad (4)$$

The idea of the proof of this theorem is based on a generalization of one-dimensional method (see [10,11]) and on some properties of starlike mappings on the unit ball of  $\mathbb{C}^n$  (see, e.g., [16,3,4,8]) and on the unit ball of a complex Banach space (see for instance [17,18,13]). We will need the following lemma.

**Lemma.** Let  $\alpha \in (0, 1)$  and  $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$  be a locally biholomorphic mapping normalized by the conditions  $f(0) = 0$ ,  $Df(0) = I$ . Then the following statements are equivalent:

- (i)  $f$  maps  $\mathbb{B}^n$  onto  $f(\mathbb{B}^n)$  biholomorphically and satisfies relation (4) for every  $w \in f(\mathbb{B}^n)$ ;
- (ii) for every  $r \in (0, 1)$ ,  $f$  maps the ball  $\mathbb{B}_r^n = r\mathbb{B}^n$  onto  $f(\mathbb{B}_r^n)$  biholomorphically and for every  $w \in f(\mathbb{B}_r^n)$  there holds the inclusion

$$w\mathbb{E}_\alpha \subset f(\mathbb{B}_r^n). \quad (5)$$

**Proof.** We start with the proof of the part (i)  $\Rightarrow$  (ii) of the lemma. Let us fix  $r \in (0, 1)$ . Obviously,  $f$  is biholomorphic in  $\mathbb{B}_r^n$ . It remains to show inclusion (5). To do it, let us consider the mapping

$$q(z) = f^{-1}(\zeta f(z)), \quad z \in \mathbb{B}^n,$$

where  $\zeta \in \mathbb{E}_\alpha$  is arbitrarily fixed. Condition (i) implies that  $q$  maps holomorphically  $\mathbb{B}^n$  into itself and  $q(0) = 0$ . Thus, by an  $n$ -dimensional version of the Schwarz Lemma (see, e.g., [14, Chapter 8]),

$$q(\mathbb{B}_r^n) \subset \mathbb{B}_r^n$$

for  $r \in (0, 1)$ . Hence, using the definition of the mapping  $q$ , we obtain

$$\zeta f(\mathbb{B}_r^n) \subset f(q(\mathbb{B}_r^n)) \subset f(\mathbb{B}_r^n).$$

From this, by the arbitrariness of  $\zeta \in \mathbb{E}_\alpha$ , we get relation (5).

Now we will prove that (ii)  $\Rightarrow$  (i). Since

$$\mathbb{B}^n = \bigcup_{r \in (0, 1)} \mathbb{B}_r^n,$$

we obtain the injectivity of  $f$  in  $\mathbb{B}^n$  and consequently, the biholomorphicity of  $f$  in  $\mathbb{B}^n$  (see, e.g., [9, Chapter 10]). Inclusion (4) follows from inclusion (5) and the equality

$$f(\mathbb{B}^n) = \bigcup_{r \in (0,1)} f(\mathbb{B}_r^n).$$

Thus the proof of the lemma is complete.  $\square$

Now we will give the proof of the main theorem.

**Proof of Theorem.** First we will prove the implication (ii)  $\Rightarrow$  (i).

Let us fix  $\alpha \in (0, 1)$  and  $z \in \mathbb{B}^n \setminus \{0\}$ . Then  $z \in \partial(\mathbb{B}_r^n)$  for some  $r \in (0, 1)$ . We will show that

$$f(z)\mathbb{E}_\alpha \subset \overline{f(\mathbb{B}_r^n)}. \quad (6)$$

For an arbitrarily fixed point  $w \in f(z)\mathbb{E}_\alpha$ , there exists  $\zeta \in \mathbb{E}_\alpha$  such that  $w = \zeta f(z)$ . Since  $f(z) \in \partial f(\mathbb{B}_r^n) = f(\partial(\mathbb{B}_r^n))$  (the mapping  $f$  is biholomorphic), there exists a sequence of points  $w_k \in f(\mathbb{B}_r^n)$ ,  $k \in \mathbb{N}$ , convergent to  $f(z)$ . By the lemma,  $\zeta w_k \in f(\mathbb{B}_r^n)$  for every  $k \in \mathbb{N}$ . Hence, using  $\lim_{k \rightarrow \infty} \zeta w_k = \zeta f(z) = w$ , we get  $w \in \overline{f(\mathbb{B}_r^n)}$ . Thus inclusion (6) is true.

Now let us denote

$$v^-(\theta) = v_{\alpha,z}^-(\theta) = f^{-1}(f(z)\rho_\alpha(\theta)e^{i\theta}), \quad \theta \in \left[(\alpha-1)\frac{\pi}{2}, 0\right],$$

where  $\rho_\alpha$  is defined in the interval  $[(\alpha-1)\frac{\pi}{2}, 0]$  by formula (3). Then, by inclusion (6),  $v^-(\theta) \in \overline{\mathbb{B}_r^n}$ , for  $\theta \in [(\alpha-1)\frac{\pi}{2}, 0]$ , i.e.,

$$\|v^-(\theta)\| \leq r = \|z\| = \|v^-(0)\|, \quad \theta \in \left[(\alpha-1)\frac{\pi}{2}, 0\right].$$

Using the above relations we obtain

$$\begin{aligned} 0 &\leq \lim_{\theta \rightarrow 0^-} \frac{\|v^-(\theta)\| - \|v^-(0)\|}{\theta} = \frac{\partial}{\partial \theta} \|v^-(\theta)\|_{\theta=0} \\ &= \frac{\partial}{\partial \theta} \sqrt{\langle v^-(\theta), v^-(\theta) \rangle} \Big|_{\theta=0} = \frac{1}{2\|v^-(0)\|} \frac{\partial}{\partial \theta} \langle v^-(\theta), v^-(\theta) \rangle \Big|_{\theta=0} \\ &= \frac{1}{\|z\|} \operatorname{Re} \left\langle \frac{\partial v^-(\theta)}{\partial \theta} \Big|_{\theta=0}, z \right\rangle = \frac{1}{\|z\|} \operatorname{Re} \left\langle (Df(z))^{-1} f(z) \frac{\partial \zeta}{\partial \theta} \Big|_{\theta=0}, z \right\rangle \\ &= \frac{1}{\|z\| \cos(\alpha \frac{\pi}{2})} \operatorname{Re} \langle (Df(z))^{-1} f(z) i e^{-i\alpha \frac{\pi}{2}}, z \rangle. \end{aligned}$$

Therefore, by arbitrariness of  $z \in \mathbb{B}^n \setminus \{0\}$ , the following inequality

$$\operatorname{Re} \langle (Df(z))^{-1} f(z) i e^{-i\alpha \frac{\pi}{2}}, z \rangle \geq 0$$

holds for  $z \in \mathbb{B}^n \setminus \{0\}$ . Hence, by Lemma 3 from [18], we get

$$\operatorname{Re} \langle i e^{-i\alpha \frac{\pi}{2}} ((Df(z))^{-1} f(z), z) \rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\},$$

and thus,

$$-\pi + \alpha \frac{\pi}{2} < \arg \langle (Df(z))^{-1} f(z), z \rangle < \alpha \frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}. \quad (7)$$

Now applying similar considerations to the function

$$v^+(\theta) = v_{\alpha,z}^+(\theta) = f^{-1}(f(z)\rho_\alpha(\theta)e^{i\theta}), \quad \theta \in \left[0, (1-\alpha)\frac{\pi}{2}\right],$$

with  $\rho_\alpha$  defined in the interval  $[0, (1-\alpha)\frac{\pi}{2}]$  by formula (3), we obtain

$$\operatorname{Re} \langle i e^{i\alpha \frac{\pi}{2}} ((Df(z))^{-1} f(z), z) \rangle < 0$$

for  $z \in \mathbb{B}^n \setminus \{0\}$ . Thus

$$-\alpha \frac{\pi}{2} < \arg \langle (Df(z))^{-1} f(z), z \rangle < \pi - \alpha \frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

Hence and by (7), we get

$$\max\left\{-\pi + \alpha \frac{\pi}{2}, -\alpha \frac{\pi}{2}\right\} < \arg\langle (Df(z))^{-1} f(z), z \rangle < \min\left\{\alpha \frac{\pi}{2}, \pi - \alpha \frac{\pi}{2}\right\}$$

for  $z \in \mathbb{B}^n \setminus \{0\}$ . Consequently,

$$-\alpha \frac{\pi}{2} < \arg\langle (Df(z))^{-1} f(z), z \rangle < \alpha \frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

Therefore  $f$  is strongly starlike of order  $\alpha \in (0, 1)$ .

Now we prove the implication (i)  $\Rightarrow$  (ii). We start with the fact that for every arbitrarily fixed  $\alpha \in (0, 1)$ ,

$$\mathbb{E}_\alpha = [0, 1] \cup \bigcup_{\delta \in [\alpha, 1)} (\Gamma_\delta^- \cup \Gamma_\delta^+),$$

where  $\Gamma_\delta^-$ ,  $\Gamma_\delta^+$  are the circular arcs given by the following polar equations

$$\zeta = \rho_\delta^-(\theta) e^{i\theta}, \quad \theta \in \left((\delta - 1)\frac{\pi}{2}, 0\right],$$

$$\zeta = \rho_\delta^+(\theta) e^{i\theta}, \quad \theta \in \left[0, (1 - \delta)\frac{\pi}{2}\right)$$

with

$$\rho_\delta^-(\theta) = \left(\cos\left(\delta \frac{\pi}{2}\right)\right)^{-1} \cos\left(\theta - \delta \frac{\pi}{2}\right), \quad \theta \in \left((\delta - 1)\frac{\pi}{2}, 0\right],$$

$$\rho_\delta^+(\theta) = \left(\cos\left(\delta \frac{\pi}{2}\right)\right)^{-1} \cos\left(\theta + \delta \frac{\pi}{2}\right), \quad \theta \in \left[0, (1 - \delta)\frac{\pi}{2}\right),$$

respectively (see [10,11]).

Let us fix  $\delta \in [\alpha, 1)$ ,  $z \in \mathbb{B}^n \setminus \{0\}$  and define the number

$$\theta^- = \theta_{\delta, z}^- = \inf\left\{\theta \in \left((\delta - 1)\frac{\pi}{2}, 0\right] : \{f(z)\rho_\delta^-(t)e^{it} : t \in (\theta, 0]\} \subset f(\mathbb{B}^n)\right\}.$$

Then  $\theta^- < 0$ , because  $\lim_{t \rightarrow 0^-} f(z)\rho_\delta^-(t)e^{it} = f(z)$  and  $f$  is biholomorphic (see the introduction).

We will show that  $\theta^- = (\delta - 1)\frac{\pi}{2}$ . To do this, in the first step we will prove that for every  $\theta \in (\theta^-, 0]$ , the norm of the function

$$u^-(t) = u_{\delta, z}^-(t) = f^{-1}(f(z)\rho_\delta^-(t)e^{it}), \quad t \in (\theta, 0],$$

is increasing. Indeed, similarly to the proof of the implication (i)  $\Rightarrow$  (ii), we obtain that for  $t \in (\theta, 0]$ ,

$$\frac{\partial}{\partial t} \|u^-(t)\| = c \operatorname{Re}\langle (Df(u^-(t)))^{-1} f(u^-(t)) i e^{i(t - \delta \frac{\pi}{2})}, u^-(t) \rangle,$$

where  $c = (\|u^-(t)\| \cos(t - \delta \frac{\pi}{2}))^{-1}$ . Since  $f$  is strongly starlike of order  $\alpha \in (0, 1)$ ,

$$\operatorname{Re}\langle (Df(u^-(t)))^{-1} f(u^-(t)), u^-(t) \rangle \neq 0$$

(see (2)) and

$$-\alpha \frac{\pi}{2} < \arg\langle (Df(u^-(t)))^{-1} f(u^-(t)), u^-(t) \rangle < \alpha \frac{\pi}{2}$$

(see (1)). Hence and by the following inequalities,

$$\alpha \leq \delta < 1, \quad (\delta - 1)\frac{\pi}{2} \leq \theta^- \leq \theta < t \leq 0,$$

we get

$$-\frac{\pi}{2} < \arg\langle (Df(u^-(t)))^{-1} f(u^-(t)) i e^{i(t - \delta \frac{\pi}{2})}, u^-(t) \rangle < \frac{\pi}{2}.$$

Thus we have

$$\operatorname{Re}\langle (Df(u^-(t)))^{-1} f(u^-(t)) i e^{i(t - \delta \frac{\pi}{2})}, u^-(t) \rangle > 0,$$

that is,  $\frac{\partial}{\partial t} \|u^-(t)\| > 0$  for  $t \in (\theta, 0]$ . Therefore,  $\|u^-(t)\|$  increases in every interval  $(\theta, 0] \subset (\theta^-, 0]$ , and hence, in the interval  $(\theta^-, 0]$ .

The above monotonicity implies that  $\|u^-(t)\| < \|u^-(0)\| = \|z\| = r$  for  $t \in (\theta^-, 0)$ , hence

$$\|f^{-1}(f(z)\rho_\delta^-(t)e^{it})\| < r,$$

that is,

$$f^{-1}(f(z)\rho_\delta^-(t)e^{it}) \in \mathbb{B}_r^n$$

for  $t \in (\theta^-, 0)$ . Therefore,

$$f(z)\rho_\delta^-(t)e^{it} \in f(\mathbb{B}_r^n), \quad t \in (\theta^-, 0). \quad (8)$$

Now we will show the equality  $\theta^- = (\delta - 1)\frac{\pi}{2}$ . Assuming, to the contrary, that  $\theta^- > (\delta - 1)\frac{\pi}{2}$ , we obtain from (8) that

$$w^- = w_{\delta,z}^- = f(z)\rho_\delta^-(\theta^-)e^{i\theta^-} \in \partial f(\mathbb{B}_r^n).$$

Thus there exists a point  $z_\delta \in \partial(\mathbb{B}_r^n)$  such that  $w^- = f(z_\delta)$ . Since  $f$  is biholomorphic (see the introduction), the ball with the center  $w^-$  and a small radius  $\varepsilon > 0$  is included also in  $f(\mathbb{B}^n)$ . Consequently, the following inclusion

$$\{f(z)\rho_\delta^-(t)e^{it} : t \in (\theta^-(\varepsilon), \theta^-)\} \subset f(\mathbb{B}^n)$$

holds for a number  $\theta^-(\varepsilon) < \theta^-$ . Therefore

$$\{f(z)\rho_\delta^-(t)e^{it} : t \in (\theta^-(\varepsilon), 0]\} \subset f(\mathbb{B}^n),$$

which is impossible, in view of the definition of the number  $\theta^-$ . Thus our supposition was false and  $\theta^- = (\delta - 1)\frac{\pi}{2}$ . Consequently,  $f(z)\rho_\delta^-(t)e^{it} \in f(\mathbb{B}^n)$  for every  $t \in ((\delta - 1)\frac{\pi}{2}, 0]$ .

Similarly, we prove that if we define the number

$$\theta^+ = \theta_{\delta,z}^+ = \sup \left\{ \theta \in \left[ 0, (1 - \delta)\frac{\pi}{2} \right) : \{f(z)\rho_\delta^+(t)e^{it} : t \in [0, \theta)\} \subset f(\mathbb{B}^n) \right\},$$

then  $\theta^+ = (1 - \delta)\frac{\pi}{2}$  and consequently,  $f(z)\rho_\delta^+(t)e^{it} \in f(\mathbb{B}^n)$  for every  $t \in [0, (1 - \delta)\frac{\pi}{2}]$ .

Since also  $f(z)[0, 1] \subset f(\mathbb{B}^n)$ , i.e.,  $f$  is starlike (see the introduction), we conclude that  $f(z)\mathbb{E}_\alpha \subset f(\mathbb{B}^n)$ .

This proves the implication (i)  $\Rightarrow$  (ii), because  $z$  was arbitrary.

The proof of the theorem is now complete.  $\square$

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